

Inverzna matrica

Transponovanu matricu matrice A obilježavamo sa A^T .
 Kofaktor A_{ij} , matrice A , elementa a_{ij} je determinanta pomnožena sa $(-1)^{i+j}$ čiji su elementi svi elementi iz matrice A osim one kolone i one vrste u kojoj se nalazi koeficijent a_{ij} .

Npr. $A = \begin{bmatrix} 3 & 7 & 2 \\ 6 & 8 & 9 \\ 1 & 2 & 4 \end{bmatrix}$, $A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 7 \\ 1 & 2 \end{vmatrix}$, $A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$, $A_{31} = (-1)^{3+1} \begin{vmatrix} 7 & 2 \\ 8 & 9 \end{vmatrix}$

$A^T = \begin{bmatrix} 3 & 6 & 1 \\ 7 & 8 & 2 \\ 2 & 9 & 4 \end{bmatrix}$ Kofaktor matrica (A_{kof}) kvadratne matrice A je matrica kofaktora A_{ik} elemenata a_{ik} dane matrice.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A_{kof} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Za matricu A kažemo da je regularna ako je $\det A \neq 0$.
 Inverznu matricu računamo po formuli

$$A^{-1} = \frac{1}{\det A} \cdot A_{kof}^T$$

Neke osobine inverzne matrice:
 $A^{-1} \cdot A = A \cdot A^{-1} = I$
 $(AB)^{-1} = B^{-1} A^{-1}$

1.) Nadi inverznu matricu matrice $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Rj: $A^{-1} = \frac{1}{\det A} \cdot A_{kof}^T$

$$\det A = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \xrightarrow{\|k_2 - \|k_1} \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$A_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \quad A_{13} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \quad A_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

$$A_{12} = (-1)^3 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1 \quad A_{21} = (-1)^3 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = 2 \quad A_{23} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$$A_{31} = (-1)^4 \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$A_{kof} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

proveraj:

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

inverzna matrica matrice A

2.) Nadi inverznu matricu matrice $B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix}$.

Rj: $B^{-1} = \frac{1}{\det B} B_{kof}^T$

$$\det B = \begin{vmatrix} 3 & 2 & 4 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{vmatrix} \xrightarrow{\|k_2 - \|k_1} \begin{vmatrix} 3 & -1 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \xrightarrow{\|k_2 - \|k_3} \begin{vmatrix} 3 & -1 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix} = 2$$

$$B_{11} = (-1)^2 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 2$$

$$B_{21} = (-1)^3 \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 0$$

$$B_{31} = (-1)^4 \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = -4$$

$$B_{12} = (-1)^3 \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 0$$

$$B_{22} = (-1)^4 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 2$$

$$B_{32} = (-1)^5 \begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4$$

$$B_{13} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$$

$$B_{23} = (-1)^5 \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

$$B_{33} = (-1)^6 \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5$$

$$B_{kof}^T = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 2 & -4 \\ -1 & -1 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 & -4 \\ 0 & 2 & -4 \\ -1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$$
 tražena inverzna matrica

3.) Nadi inverznu matricu matrice $C = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$.

Rj: $C^{-1} = \frac{1}{\det C} C_{kof}^T$

$$\det C = \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} = 3$$

$$C_{11} = (-1)^2 \cdot 4 = 4$$

$$C_{21} = (-1)^3 \cdot 1 = -1$$

$$C_{kof}^T = \begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{5}{3} & \frac{2}{3} \end{bmatrix}$$

4.) Nadi inverznu matricu sljedećih matrica:

a) $A = \begin{bmatrix} 3 & 4 & 4 \\ 1 & 6 & 1 \\ 2 & 3 & 3 \end{bmatrix}$

c) $C = \begin{bmatrix} 7 & 3 & 3 \\ 6 & 3 & 4 \\ -1 & -2 & -3 \end{bmatrix}$

Rješenje:

a) $A^{-1} = \begin{bmatrix} \frac{3}{5} & 0 & -4 \\ -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{14}{5} \end{bmatrix}$

b) $B = \begin{bmatrix} -3 & -1 & -1 \\ 1 & 3 & 2 \\ -2 & -1 & -2 \end{bmatrix}$

c) $\det C = 8$

b) $B^{-1} = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \end{bmatrix}$

Matricne jednačine

U sljedećim primjerima neka su A, B, C, X neke date kvadratne matrice.

$$A^{-1} \cdot B \neq B \cdot A^{-1}$$

$$A \cdot B \neq B \cdot A$$

Matrice se ne mogu dijeliti.

Da bismo odredili nepoznatu X u matricnoj jednačini prvobitno trebamo izvesti formulu za nepoznatu X .

$$\# A \cdot X = B \quad / \cdot A^{-1} \text{ sa lijeve strane}$$

$$A^{-1} \cdot A \cdot X = A^{-1} \cdot B$$

$$1 \cdot X = A^{-1} \cdot B$$

$$X = A^{-1} \cdot B$$

$$\# X^{-1} \cdot A = B^{-1} \quad / \cdot A^{-1} \text{ sa desne strane}$$

$$X^{-1} \cdot A \cdot A^{-1} = B^{-1} \cdot A^{-1}$$

$$X^{-1} \cdot 1 = B^{-1} \cdot A^{-1}$$

$$X^{-1} = B^{-1} \cdot A^{-1} \quad / (e^{-1})$$

$$X = A \cdot B$$

$$\# A \cdot X \cdot B = C \quad / \cdot A^{-1} \text{ sa lijeve strane}$$

$$A^{-1} \cdot A \cdot X \cdot B = A^{-1} \cdot C$$

$$1 \cdot X \cdot B = A^{-1} \cdot C \quad / \cdot B^{-1} \text{ sa desne strane}$$

$$X \cdot B \cdot B^{-1} = A^{-1} \cdot C \cdot B^{-1}$$

$$X \cdot 1 = A^{-1} \cdot C \cdot B^{-1}$$

$$X = A^{-1} \cdot C \cdot B^{-1}$$

$$\# A \cdot X + 1 = X - 21$$

$$AX - X = -1 - 21$$

$$\underbrace{(A-1)}_B X = -31$$

$$BX = -31 \quad / \cdot B^{-1} \text{ sa desne strane}$$

$$B^{-1} \cdot B \cdot X = B^{-1} \cdot (-31)$$

$$1 \cdot X = -3B^{-1}$$

$$X = -3(A-1)^{-1}$$

$$\# A^{-1} \cdot X = X - 1$$

$$A^{-1} \cdot X - X = -1$$

$$\underbrace{(A^{-1} - 1)}_B \cdot X = -1$$

$$BX = -1 \quad / \cdot B^{-1} \text{ sa lijeve strane}$$

$$B^{-1} \cdot B \cdot X = -B^{-1} \cdot 1$$

$$X = -B^{-1}$$

$$X = -(A^{-1} - 1)^{-1}$$

$$\# \underbrace{(A+31)}_C (X-1) = B$$

$$C(X-1) = B \quad / \cdot C^{-1} \text{ sa lijeve strane}$$

$$C^{-1}C(X-1) = C^{-1} \cdot B$$

$$X-1 = C^{-1} \cdot B$$

$$X = C^{-1} \cdot B + 1$$

$$X = (A+31)^{-1} \cdot B + 1$$

$$\# B^{-1} \cdot X \cdot A = (3B+21)^{-1}$$

$$B \cdot B^{-1} \cdot X \cdot A = B(3B+21)^{-1} \quad / \cdot B \text{ sa lijeve strane}$$

$$X \cdot A = B(3B+21)^{-1} \quad / \cdot A^{-1} \text{ sa desne strane}$$

$$X = B(3B+21)^{-1} \cdot A^{-1}$$

$$\# (AXB)^{-1} = B^{-1}(X^{-1}+B) \quad / \cdot (AXB) \text{ sa lijeve strane}$$

$$(AXB)(AXB)^{-1} = AX \underbrace{B B^{-1}}_1 (X^{-1}+B)$$

$$1 = AX(X^{-1}+B)$$

$$1 = AX X^{-1} + AXB$$

$$1 = A + AXB$$

$$AXB = 1 - A \quad / \cdot A^{-1} \text{ sa lijeve strane}$$

$$A^{-1}AXB \cdot B^{-1} = A^{-1}(1-A) \cdot B^{-1}$$

$$X = A^{-1}(1-A) \cdot B^{-1}$$

1. Riješiti matricnu jednačinu

$$Rj: X \cdot A = B \quad / \cdot A^{-1} \text{ sa desne strane}$$

$$X = B \cdot A^{-1}, \quad A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 2 & 3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot A_{kof}^T$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 2 & 3 & 5 \end{vmatrix} \begin{array}{l} |I_k - III_k \\ |I_k - III_k \end{array} \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & -3 \\ -3 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -3 & -2 \end{vmatrix} = -1$$

$$A_{31} = (-1)^4 \begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} = -1$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} = 2$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -1$$

$$A_{kof}^T = \begin{bmatrix} -1 & -2 & -1 \\ -1 & 3 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & -2 \\ -1 & 1 & 1 \end{bmatrix}$$

$$X \cdot \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_{11} = (-1)^2 \begin{vmatrix} -2 & -3 \\ 3 & 5 \end{vmatrix} = -1$$

$$A_{12} = (-1)^3 \begin{vmatrix} -1 & -3 \\ 2 & 5 \end{vmatrix} = -1$$

$$A_{13} = (-1)^4 \begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} = 1$$

$$A_{21} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} = 2$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} = 3$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$$X = B \cdot A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & -2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -2 \\ 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \text{ riješi je matricne jednačine}$$

2) Riješiti matricnu jednačinu $A \cdot X = X + I$ ako je $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & -2 \\ 3 & 1 & -1 \end{bmatrix}$.

Rj: $A \cdot X = X + I$ $C = A - I = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & -2 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & -2 \\ 3 & 1 & -2 \end{bmatrix}$

$$A \cdot X - X = I$$

$$(A - I) \cdot X = I \quad | \cdot (A - I)^{-1} \text{ sa lijeve strane}$$

$$C^{-1} = \frac{1}{\det C} C_{\text{kof}}^T$$

$$(A - I)(A - I)^{-1} X = (A - I)^{-1} \cdot I$$

$$X = (A - I)^{-1}$$

$$\det C = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 2 & -2 \\ 3 & 1 & -2 \end{vmatrix} \xrightarrow{I_k + III_k} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 3 & -1 & -2 \end{vmatrix}$$

$$C_{11} = (-1)^2 \begin{vmatrix} 2 & -2 \\ 1 & -2 \end{vmatrix} = -2$$

$$C_{21} = (-1)^3 \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = -1$$

$$C_{31} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} = 0$$

$$C_{12} = (-1)^3 \begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix} = -4$$

$$C_{22} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 3 & -2 \end{vmatrix} = -3$$

$$C_{32} = (-1)^5 \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 1$$

$$C_{13} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

$$C_{23} = (-1)^5 \begin{vmatrix} 0 & -1 \\ 3 & 1 \end{vmatrix} = -3$$

$$C_{33} = (-1)^6 \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 1$$

$$C_{\text{kof}}^T = \begin{bmatrix} -2 & -1 & 0 \\ -4 & -3 & 1 \\ -5 & -3 & 1 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & -1 \\ 5 & 3 & -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & -1 \\ 5 & 3 & -1 \end{bmatrix} \text{ riješi je}$$

3) Riješiti matricnu jednačinu $(A+B)^{-1} A \cdot X^{-1} = B^{-1}$ gdje su matrice $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ i $B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$.

Rj: $(A+B)^{-1} A \cdot X^{-1} = B^{-1}$ | $\cdot (A+B)$ sa lijeve strane

$$X^{-1} = A^{-1} (A+B) \cdot B^{-1} \quad | \cdot A$$

$$(A+B)(A+B)^{-1} A \cdot X^{-1} = (A+B) \cdot B^{-1}$$

$$A \cdot X^{-1} = (A+B) \cdot B^{-1} \quad | \cdot A^{-1} \text{ sa lijeve strane}$$

$$X = B (A+B)^{-1} A$$

$$A^{-1} \cdot A \cdot X^{-1} = A^{-1} (A+B) \cdot B^{-1}$$

$$C = A+B = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}, \quad C^{-1} = \frac{1}{\det C} C_{\text{kof}}^T, \quad \det C = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix} = 13, \quad \begin{aligned} C_{11} &= (-1)^2 \cdot 3 = 3 \\ C_{12} &= (-1)^3 \cdot 1 = -1 \\ C_{21} &= (-1)^2 \cdot (-1) = -1 \\ C_{22} &= (-1)^3 \cdot 4 = -4 \end{aligned}$$

$$C_{\text{kof}}^T = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} \frac{3}{13} & \frac{1}{13} \\ -\frac{1}{13} & \frac{4}{13} \end{bmatrix}$$

$$X = B \cdot C^{-1} \cdot A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \cdot \frac{1}{13} \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 7 & 5 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 15 & -6 \\ 5 & 12 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{15}{13} & -\frac{6}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix} \text{ riješi je matricne jednačine}$$

4) Riješiti matricnu jednačinu $(A+3I)(X-I) = B$, ako je $A = \begin{bmatrix} -2 & 5 & -2 \\ 2 & 8 & 0 \\ -1 & -5 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 21 & 1 \\ 2 & 50 & -2 \\ 1 & -22 & 0 \end{bmatrix}$ i I jedinična matrica.

Rj: $(A+3I)(X-I) = B$ | $\cdot (A+3I)^{-1}$ sa lijeve strane

$$C^{-1} = \frac{1}{\det C} C_{\text{kof}}^T$$

$$(A+3I)^{-1} (A+3I)(X-I) = (A+3I)^{-1} \cdot B$$

$$X - I = (A+3I)^{-1} \cdot B$$

$$X = (A+3I)^{-1} \cdot B + I$$

$$\det C = \begin{vmatrix} 1 & 5 & -2 \\ 2 & 11 & 0 \\ -1 & -5 & 1 \end{vmatrix} \xrightarrow{I_k + III_k} \begin{vmatrix} 0 & 0 & 1 \\ 2 & 11 & 0 \\ -1 & -5 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2 & 11 & 0 \\ -1 & -5 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 11 \\ -1 & -5 \end{vmatrix} = -1$$

$$C = A + 3I = \begin{bmatrix} -2 & 5 & -2 \\ 2 & 8 & 0 \\ -1 & -5 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & -2 \\ 2 & 11 & 0 \\ -1 & -5 & 1 \end{bmatrix}$$

$$C_{11} = (-1)^2 \begin{vmatrix} 11 & 0 \\ -5 & 1 \end{vmatrix} = 11$$

$$C_{21} = (-1)^3 \begin{vmatrix} 5 & -2 \\ -5 & 1 \end{vmatrix} = 5$$

$$C_{31} = (-1)^4 \begin{vmatrix} 5 & -2 \\ 11 & 0 \end{vmatrix} = 22$$

$$C_{12} = (-1)^3 \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = -2$$

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = -1$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & -2 \\ 2 & 0 \end{vmatrix} = -4$$

$$C_{13} = (-1)^4 \begin{vmatrix} 2 & 11 \\ -1 & -5 \end{vmatrix} = 1$$

$$C_{23} = (-1)^5 \begin{vmatrix} 1 & 5 \\ -1 & -5 \end{vmatrix} = 0$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & 5 \\ 2 & 11 \end{vmatrix} = 1$$

$$C_{\text{kof}}^T = \begin{bmatrix} 11 & 5 & 22 \\ -2 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} -11 & -5 & -22 \\ 2 & 1 & 4 \\ -1 & 0 & -1 \end{bmatrix}$$

$$C^{-1} \cdot B = \begin{bmatrix} -11 & -5 & -22 \\ 2 & 1 & 4 \\ -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 21 & 1 \\ 2 & 50 & -2 \\ 1 & -22 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

$$X = (A+3I)^{-1} \cdot B + I = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & 0 \\ 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 0 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{rešene matricne jednačine}$$

5) Riješiti matricnu jednačinu $(X^{-1} + B^{-1})^{-1} = AX$ ako su

$$A = \begin{bmatrix} 3 & 3 & 2 \\ -4 & 1 & -4 \\ -3 & 1 & -3 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

R: $(X^{-1} + B^{-1})^{-1} = AX$ / $(X^{-1} + B^{-1})$ sa desne strane

$$(X^{-1} + B^{-1}) \cdot (X^{-1} + B^{-1}) = AX(X^{-1} + B^{-1})$$

$$I = A + AXB^{-1}$$

$$AXB^{-1} = I - A \quad / \quad A^{-1} \text{ sa lijeve str. } B \text{ sa desne str.}$$

$$A^{-1} \cdot A \cdot X \cdot B^{-1} \cdot B = A^{-1}(I - A) \cdot B$$

$$X = A^{-1}(I - A) \cdot B$$

$$A_{21} = (-1)^3 \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} = 11$$

$$A_{31} = (-1)^4 \begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} = -14$$

$$A_{22} = (-1)^4 \begin{vmatrix} 3 & 2 \\ -3 & -3 \end{vmatrix} = -3$$

$$A_{32} = (-1)^5 \begin{vmatrix} 2 & 2 \\ -4 & -4 \end{vmatrix} = 4$$

$$A_{23} = (-1)^5 \begin{vmatrix} 3 & 3 \\ -3 & 1 \end{vmatrix} = -12$$

$$A_{33} = (-1)^6 \begin{vmatrix} 3 & 3 \\ -4 & 1 \end{vmatrix} = 15$$

$$A^{-1} = \begin{bmatrix} 1 & 11 & -14 \\ 0 & -3 & 4 \\ -1 & -12 & 15 \end{bmatrix}$$

$$X = A^{-1}(I - A) \cdot B = \begin{bmatrix} 1 & 11 & -14 \\ 0 & -3 & 4 \\ -1 & -12 & 15 \end{bmatrix} \begin{bmatrix} -2 & -3 & -2 \\ 4 & 0 & 4 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 11 & -14 \\ 0 & -3 & 4 \\ -1 & -12 & 15 \end{bmatrix} \begin{bmatrix} -5 & 8 & -6 \\ 4 & -4 & 12 \\ 2 & -2 & 10 \end{bmatrix} = \begin{bmatrix} 11 & -8 & -14 \\ -4 & 4 & 4 \\ -13 & 10 & 12 \end{bmatrix} \quad \text{rešene matricne jednačine}$$

6) Riješiti matricnu jednačinu:

$$X \cdot \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 4 \\ 3 & 4 & 2 \end{bmatrix} = X^{-1} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

Ako označimo $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ imamo

$$XA + B = X^{-1}B$$

$$XA - X^{-1}B = -B$$

$$X(A - B) = -B \quad / \quad (A - B)^{-1} \text{ sa lijeve strane}$$

$$X = -B(A - B)^{-1}$$

$$\det C = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} \begin{matrix} |k+1|_k \\ |k+1|_k \\ |k+1|_k \end{matrix} \begin{vmatrix} 0 & -1 & -1 \\ 2 & 1 & -1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= (-2) \begin{vmatrix} -1 & -1 \\ -1 & 0 \end{vmatrix} = (-2) \cdot (-1) = 2$$

$$C = A - B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$C_{11} = (-1)^2 \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = -1$$

$$C_{21} = 1 \quad C_{31} = 2$$

$$C_{22} = 1 \quad C_{32} = 0$$

$$C_{12} = (-1)^3 \cdot 1 = -1$$

$$C_{23} = 0 \quad C_{33} = 2$$

$$C_{13} = -2$$

$$C^{-1} = \frac{1}{\det C} \cdot C_{kof}^T$$

$$C^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

$$X = -B \cdot C^{-1} = - \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 4 \\ 3 & 4 & 2 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

$$X = -\frac{1}{2} \begin{bmatrix} -15 & 5 & 12 \\ -9 & 1 & 8 \\ -11 & 7 & 10 \end{bmatrix} = \begin{bmatrix} \frac{15}{2} & -\frac{5}{2} & -6 \\ \frac{9}{2} & -\frac{1}{2} & -4 \\ \frac{11}{2} & -\frac{7}{2} & -5 \end{bmatrix} \quad \text{rešene matricne jednačine}$$

7) Riješiti matricnu jednačinu $(A + X)(B - 2I) = A$, ako su

$$A = \begin{bmatrix} -2 & -3 & -4 \\ 1 & -2 & 3 \\ 4 & 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}, \quad I \text{ jedinična matrica.}$$

8) Riješiti matricnu jednačinu $A^{-1}X + B = AX$, ako su

$$A = \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$$

9) Riješiti matricnu jednačinu $(XB^{-1})^{-1} = X^{-1} + A$, ako su

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 1 & 2 & 3 \\ 0 & 3 & 2 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Rješenja:

$$7. \quad X = \begin{bmatrix} -2 & 10 & -1 \\ 2 & 2 & -5 \\ -6 & -14 & 19 \end{bmatrix}$$

$$8. \quad X = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$$

$$9. \quad X = \begin{bmatrix} 3 & -\frac{3}{2} & -\frac{17}{2} \\ 1 & -1 & -5 \\ 0 & \frac{5}{2} & \frac{15}{2} \end{bmatrix}$$

Data je matricna jednačina $A(X-B)^{-1} = B^{-1}A$ i matrice

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix}.$$

a) koji uslov moraju zadovoljavati matrice A i B da bi data jednačina imala rješenje $X=2B$?

b) Riješiti datu jednačinu ako matrice A i B ne zadovoljavaju uslov dobijen pod a)

Rj. a) $A(X-B)^{-1} = B^{-1}A$

$$X = 2B$$

$A \cdot B^{-1} = B^{-1}A$ uslov koji moraju zadovoljavati matrice A i B da bi data matrica imala rješenje $X=2B$.

Usluv možemo provjeriti i na drugi način:

$$A = B^{-1}AB$$

ili

$$B = A^{-1}B \cdot A$$

b) $A(X-B)^{-1} = B^{-1}A$ $| \cdot (X-B)$ sa desne str.

$$B^{-1}A(X-B) = A \quad | \cdot B$$
 sa lijeve str.

$$A(X-B) = BA \quad | \cdot A^{-1}$$
 sa lijeve str.

$$X-B = A^{-1}BA$$

$$X = A^{-1}BA + B$$

i odatudje možemo provjeriti uslov koji smo dobili pod a) (ako je $B = A^{-1}BA$ tada jednačina ima rješenje $X=2B$)

$$A_{11} = (-1)^2 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \quad A_{21} = (-1)^3 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -2 \quad A_{31} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \quad A_{\text{kof}} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A_{12} = (-1)^3 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0 \quad A_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \quad A_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$A_{13} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} = 0 \quad A_{23} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = 1 \quad A_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \quad A_{\text{kof}}^T = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = 2 \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & 0 & 4 \end{bmatrix}$$

$$A^{-1} \cdot B \cdot A = 2 \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & 0 & 4 \end{bmatrix} = 2 \begin{bmatrix} -2 & -2 & -2 \\ -1 & 3 & -1 \\ 5 & 3 & 7 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{5}{2} & \frac{3}{2} & \frac{7}{2} \end{bmatrix}$$

$$X = A^{-1}BA + B = \begin{bmatrix} -1 & -1 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{5}{2} & \frac{3}{2} & \frac{7}{2} \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ \frac{3}{2} & \frac{5}{2} & -\frac{1}{2} \\ \frac{11}{2} & \frac{1}{2} & \frac{11}{2} \end{bmatrix}$$

rješenje matricne jednačine

Riješiti matricnu jednačinu $X \cdot A^{-1} = B^{-1}$ ako su

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}.$$

Rj.

$$X \cdot A^{-1} = B^{-1} \quad | \cdot A$$
 sa desne strane

$$\underbrace{X \cdot A^{-1} \cdot A}_{I} = B^{-1} \cdot A$$

$$X = B^{-1} \cdot A$$

$$B^{-1} = \frac{1}{\det B} B_{\text{kof}}^T$$

$$\det B = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{vmatrix} \xrightarrow{|_2 - |_1}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 0+1$$

$$\det B = 1$$

$$B_{11} = (-1)^2 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$B_{21} = (-1)^3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1$$

$$B_{31} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = -1$$

$$B_{12} = (-1)^3 \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} = 0$$

$$B_{22} = (-1)^4 \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1$$

$$B_{32} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = 2$$

$$B_{13} = (-1)^4 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

$$B_{23} = (-1)^5 \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = -1$$

$$B_{33} = (-1)^6 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0$$

$$B_{\text{kof}} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ -1 & 2 & 0 \end{bmatrix},$$

$$B_{\text{kof}}^T = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix},$$

$$B^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$X = B^{-1} \cdot A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 7 \\ 5 & 4 & -7 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\begin{array}{lll} 2-3-1 & 0+3+2 & 2-3+0 \\ 3-2-1 & 0+2+2 & 3-2+0 \\ 4-1+4 & 0+1-8 & 4-1+0 \end{array}$$

$$X = \begin{bmatrix} -2 & 0 & 7 \\ 5 & 4 & -7 \\ -1 & 1 & 3 \end{bmatrix}$$

traženo rješenje

Riješiti matricnu jednačinu $X^{-1}AB = B^{-1}A^{-1}$,

$$A = \begin{bmatrix} 1 & 1 & 6 \\ 2 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -4 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Rj: $X^{-1}AB = B^{-1}A^{-1}$

$$X^{-1}AB = (AB)^{-1} \quad / (AB)^{-1} \text{ sa desne strane}$$

$$X^{-1} = (AB)^{-1} (AB)^{-1}$$

$$X = (AB) \cdot (AB)$$

$$X = (AB)^2$$

$$A \cdot B = \begin{bmatrix} 1 & 1 & 6 \\ 2 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & -4 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 13 \\ 1 & 10 & -3 \\ 2 & -4 & 3 \end{bmatrix}$$

$\begin{matrix} 2+1+6 & 4-3+0 & 0+1+1 \\ -1+0 & -2+12+0 & 0-4+0 \\ 0+1+12 & 0-3+0 & 0+1+2 \end{matrix}$

$$(AB)^2 = \begin{bmatrix} 9 & -5 & 13 \\ 1 & 10 & -3 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 9 & -5 & 13 \\ 1 & 10 & -3 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 102 & -147 & 171 \\ 13 & 107 & -26 \\ 20 & -62 & 47 \end{bmatrix}$$

$$\begin{matrix} 81-5+26 & -45-50-52 & 117+15+39 \\ 3+10-6 & -5+100+12 & 13-30-9 \\ 18-4+6 & -10-40-12 & 26+12+9 \end{matrix}$$

$$X = \begin{bmatrix} 102 & -147 & 171 \\ 13 & 107 & -26 \\ 20 & -62 & 47 \end{bmatrix}$$

Riješiti matricnu jednačinu $(A+I)^{-1} \cdot X \cdot (3A+I) = 2A$ gdje je I jedinična matrica drugog reda a

$$A = \begin{bmatrix} 7 & 8 \\ -6 & -7 \end{bmatrix}.$$

Rj: $(A+I)^{-1} \cdot X \cdot (3A+I) = 2A \quad / (A+I) \text{ sa lijeve strane}$

$$X \cdot (3A+I) = (A+I) \cdot 2A \quad / (3A+I)^{-1} \text{ sa desne strane}$$

$$X = (A+I) \cdot 2A \cdot (3A+I)^{-1}$$

$$A = \begin{bmatrix} 7 & 8 \\ -6 & -7 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A+I = \begin{bmatrix} 8 & 8 \\ -6 & -6 \end{bmatrix}$$

$$3A+I = \begin{bmatrix} 22 & 24 \\ -18 & -20 \end{bmatrix}$$

$$3A = \begin{bmatrix} 21 & 24 \\ -18 & -21 \end{bmatrix}$$

Označimo sa $B = 3A+I$ pa pronadimo B^{-1}

$$B^{-1} = \frac{1}{\det B} \cdot B_{\text{kof}}^T \quad \det B = \begin{vmatrix} 22 & 24 \\ -18 & -20 \end{vmatrix} = -440 + 432 = -8$$

$$B_{11} = (-1)^2 \cdot (-20) = -20$$

$$B_{21} = (-1)^3 \cdot 24 = -24$$

$$B_{12} = (-1)^3 \cdot (-18) = 18$$

$$B_{22} = (-1)^4 \cdot 22 = 22$$

$$B_{\text{kof}} = \begin{bmatrix} -20 & 18 \\ -24 & 22 \end{bmatrix}$$

$$B^{-1} = \frac{-1}{8} \begin{bmatrix} -20 & -24 \\ 18 & 22 \end{bmatrix} = (3A+I)^{-1}$$

$$X = (A+I) \cdot 2A \cdot (3A+I)^{-1} = \begin{bmatrix} 8 & 8 \\ -6 & -6 \end{bmatrix} \cdot 2 \cdot \begin{bmatrix} 7 & 8 \\ -6 & -7 \end{bmatrix} \cdot \frac{-1}{8} \begin{bmatrix} -20 & -24 \\ 18 & 22 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 4 & 4 \\ -3 & -3 \end{bmatrix} \cdot 2 \begin{bmatrix} 7 & 8 \\ -6 & -7 \end{bmatrix} \cdot \frac{-1}{8} \cdot 2 \begin{bmatrix} -10 & -12 \\ 9 & 11 \end{bmatrix} = 8 \cdot \frac{-1}{8} \begin{bmatrix} 4 & 4 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} -10 & -12 \\ 9 & 11 \end{bmatrix}$$

$$= (-1) \begin{bmatrix} 4 & 4 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} -10 & -12 \\ 9 & 11 \end{bmatrix} = (-1) \begin{bmatrix} -4 & -4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -3 & -3 \end{bmatrix} \quad \text{rešavaj matricne jednačine}$$

(#) Riješiti matricnu jednačinu $(AXB)^{-1} = B^{-1}(X^{-1} + B)$

ako je $A = \begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

R: $(AXB)^{-1} = B^{-1}(X^{-1} + B)$

$B^{-1}X^{-1}A^{-1} = B^{-1}X^{-1} + B^{-1} \cdot B$ / B sa lijeve strane

$X^{-1}A^{-1} = X^{-1} + B$

$X^{-1}A^{-1} - X^{-1} = B$

$X^{-1}(A^{-1} - I) = B$ / $(A^{-1} - I)^{-1}$ sa desne strane

$X^{-1} = B(A^{-1} - I)^{-1}$ / -1

$X = (A^{-1} - I) \cdot B^{-1}$

$A^{-1} = \frac{1}{\det A} \cdot A_{\text{koF}}^T$

$A_{11} = (-1)^2 \begin{vmatrix} -3 & 1 \\ -5 & -1 \end{vmatrix} = 3 + 5 = 8$

$A_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -(-2 - 3) = 5$

$A_{13} = (-1)^4 \begin{vmatrix} 2 & -3 \\ 3 & -5 \end{vmatrix} = -10 + 9 = -1$

$A_{21} = (-1)^3 \begin{vmatrix} -4 & 5 \\ -5 & -1 \end{vmatrix} = -(4 + 25) = -29$ $A_{31} = 11$

$A_{22} = (-1)^4 \begin{vmatrix} 3 & 5 \\ 3 & -1 \end{vmatrix} = -3 - 15 = -18$ $A_{32} = 7$

$A_{23} = (-1)^5 \begin{vmatrix} 3 & -4 \\ 3 & -5 \end{vmatrix} = -(-15 + 12) = 3$ $A_{33} = -1$

$A_{\text{koF}} = \begin{bmatrix} 8 & 5 & -1 \\ -29 & -18 & 3 \\ 11 & 7 & -1 \end{bmatrix}$ $A^{-1} = (-1) \begin{bmatrix} 8 & -29 & 11 \\ 5 & -18 & 7 \\ -1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{bmatrix}$

$\det B = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix} \begin{matrix} \parallel v_1 - v_2 \\ \parallel v_1 - v_2 \end{matrix} \begin{vmatrix} 1 & 2 & 2 \\ 0 & -3 & -6 \\ 0 & -6 & -3 \end{vmatrix} = \begin{vmatrix} -3 & -6 \\ -6 & -3 \end{vmatrix} = 9 - 36 = -27$

$B^{-1} = \frac{1}{\det B} \cdot B_{\text{koF}}^T = \frac{(-1)}{-27} \begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} = \frac{1}{27} \cdot 3 \cdot \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$
(slučajno) VJEŽBU

$B^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$A^{-1} - I = \begin{bmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 29 & -11 \\ -5 & 17 & -7 \\ 1 & -3 & 0 \end{bmatrix}$

$X = (A^{-1} - I) \cdot B^{-1} = \begin{bmatrix} -9 & 29 & -11 \\ -5 & 17 & -7 \\ 1 & -3 & 0 \end{bmatrix} \cdot \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 27 & 33 & -87 \\ 15 & 21 & -51 \\ -5 & -1 & 8 \end{bmatrix}$

$X = \begin{bmatrix} 3 & \frac{11}{3} & -\frac{29}{3} \\ \frac{5}{3} & \frac{7}{3} & -\frac{17}{3} \\ -\frac{5}{9} & -\frac{1}{9} & \frac{8}{9} \end{bmatrix}$ riješite matricne jednačine

Riješiti matricnu jednačinu $A \cdot X^{-1} \cdot B = B \cdot A$, ako je $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ i $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

R:

$$A X^{-1} B = B \cdot A \quad / \cdot A^{-1} \text{ sa lijeve strane}$$

$$X^{-1} B = A^{-1} B \cdot A \quad / \cdot B^{-1} \text{ sa desne strane}$$

$$X^{-1} = A^{-1} B \cdot A \cdot B^{-1} \quad /^{-1}$$

$$X = B A^{-1} B^{-1} A$$

$$A^{-1} = \frac{1}{\det A} \cdot A_{\text{kof}}^T \quad \det A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \quad A_{\text{kof}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A_{11} = 1 \quad A_{21} = -1 \quad A_{12} = 0 \quad A_{22} = 1$$

$$B^{-1} = \frac{1}{\det B} \cdot B_{\text{kof}}^T \quad B_{11} = 1 \quad B_{21} = 0 \quad B_{12} = -1 \quad B_{22} = 1$$

$$B_{\text{kof}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad B_{\text{kof}}^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$B \cdot A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B^{-1} \cdot A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X = B A^{-1} B^{-1} A =$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{traženo rješenje}$$

Riješiti matricnu jednačinu: $A X - 2B = 3X + A$ gdje je

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 3 & 1 \\ 4 & 0 & 3 \end{bmatrix}$$

R: $A X - 2B = 3X + A$

$$A X - 3X = 2B + A$$

$$\underbrace{(A - 3I)}_M X = \underbrace{2B + A}_N$$

$$M X = N \quad / \cdot M^{-1} \text{ sa lijeve str.}$$

$$M^{-1} M X = M^{-1} N$$

$$X = M^{-1} N$$

$$M^{-1} = \frac{1}{\det M} M_{\text{kof}}^T$$

$$\det M = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1 = 6$$

$$M_{11} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2$$

$$M_{12} = (-1)^5 \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = -6$$

$$M_{13} = (-1)^6 \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6$$

$$M_{\text{kof}} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -6 & 6 \end{bmatrix}$$

$$M_{\text{kof}}^T = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix}$$

$$M^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix}$$

$$X = M^{-1} N = \frac{1}{6} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 4 & 5 & 0 \\ 4 & 11 & 4 \\ 8 & 0 & 10 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 20 & -1 & 16 \\ -36 & 33 & 48 \\ 48 & 0 & 60 \end{bmatrix}$$

$$8 - 4 + 16$$

$$10 - 11 + 0$$

$$0 - 4 + 20$$

$$0 + 12 - 48$$

$$0 + 32 + 0$$

$$12 - 60$$

$$X = \begin{bmatrix} \frac{10}{3} & -\frac{1}{6} & \frac{8}{3} \\ -6 & \frac{11}{2} & 8 \\ 8 & 0 & 10 \end{bmatrix} \quad \text{traženo rješenje}$$

$$M = A - 3I = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$N = 2B + A = \begin{bmatrix} -2 & 4 & 0 \\ 4 & 6 & 2 \\ 8 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 0 \\ 4 & 11 & 4 \\ 8 & 0 & 10 \end{bmatrix}$$

$$M_{11} = (-1)^2 \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = 2 \quad M_{21} = (-1)^3 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

$$M_{12} = (-1)^3 \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0 \quad M_{22} = (-1)^4 \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

$$M_{13} = (-1)^4 \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0 \quad M_{23} = (-1)^5 \begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

#) Riješiti matricnu jednačinu $(XA+B)^{-1}(XC+B)=C$,
 ako je $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ i $C = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

Rj: $(XA+B)^{-1}(XC+B)=C$ / $(XA+B)$ sa lijeve strane

$$\underline{(XA+B)(XA+B)^{-1}}(XC+B) = (XA+B) \cdot C$$

$$I \quad XC+B = XAC+BC$$

$$X = B(C-I)(C-AC)^{-1}$$

$$XC - XAC = BC - B$$

$$C-I = \begin{bmatrix} -2 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X(C-AC) = BC - B \quad / (C-AC)^{-1} \text{ sa desne strane}$$

$$B(C-I) = \begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 4 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 8 \\ 0 & -2 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

Označimo sa $D = C - AC = \begin{bmatrix} -2 & 4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

Izračunajmo D^{-1} . $D^{-1} = \frac{1}{\det D} D_{kof}^T$

$$D_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ 0 & -4 \end{vmatrix} = -4$$

$$D_{21} = (-1)^3 \begin{vmatrix} 4 & -6 \\ 0 & -4 \end{vmatrix} = 16$$

$$D_{31} = (-1)^4 \begin{vmatrix} 4 & -6 \\ 1 & 0 \end{vmatrix} = 6$$

$$D_{12} = (-1)^2 \begin{vmatrix} 0 & 0 \\ 0 & -4 \end{vmatrix} = 0$$

$$D_{22} = (-1)^4 \begin{vmatrix} -2 & -6 \\ 0 & -4 \end{vmatrix} = 8$$

$$D_{32} = (-1)^5 \begin{vmatrix} -2 & -6 \\ 6 & 0 \end{vmatrix} = 0$$

$$D_{13} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$$D_{23} = (-1)^5 \begin{vmatrix} -2 & 4 \\ 0 & 0 \end{vmatrix} = 0$$

$$D_{33} = (-1)^6 \begin{vmatrix} -2 & 4 \\ 0 & 1 \end{vmatrix} = -2$$

$$\det D = \begin{vmatrix} -2 & 4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{vmatrix} = (-4) \begin{vmatrix} -2 & 4 \\ 0 & 1 \end{vmatrix} = 8 \quad D_{kof} = \begin{bmatrix} -4 & 0 & 0 \\ 16 & 8 & 0 \\ 6 & 0 & -2 \end{bmatrix} \quad D_{kof}^T = \begin{bmatrix} -4 & 16 & 6 \\ 0 & 8 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} -\frac{1}{2} & 2 & \frac{3}{4} \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}, \quad X = B(C-I)(C-AC)^{-1} = \frac{1}{8} \begin{bmatrix} -4 & 4 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 16 & 6 \\ 0 & 8 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$X = \frac{1}{8} \begin{bmatrix} 16 & -32 & -30 \\ 0 & 16 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -\frac{15}{4} \\ 0 & 2 & \frac{1}{4} \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} \text{ traženo rješenje}$$

#) Riješiti matricnu jednačinu $XAB=C$, $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,
 $B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$, $C = [0 \ 4 \ 4]$.

Rj: $XAB=C$ / $(AB)^{-1}$ sa desne strane

$$X(AB)(AB)^{-1} = C \cdot (AB)^{-1}$$

$$X = C \cdot (AB)^{-1}$$

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 4 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 4 & 1 \\ -1 & 2 & 3 \end{vmatrix} = (-2) \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = (-2)(3+1) = -8$$

AB označimo sa M , nađimo M^{-1}

$$M_{11} = (-1)^2 \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 10$$

$$M_{21} = (-1)^3 \begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix} = -6$$

$$M_{31} = (-1)^4 \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2$$

$$M_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4$$

$$M_{22} = (-1)^4 \begin{vmatrix} 0 & 0 \\ -1 & 3 \end{vmatrix} = 0$$

$$M_{32} = (-1)^5 \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$$

$$M_{13} = (-1)^4 \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = 6$$

$$M_{23} = (-1)^5 \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} = -2$$

$$M_{33} = (-1)^6 \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} = -2$$

$$M_{kof} = \begin{bmatrix} 10 & -4 & 6 \\ -6 & 0 & -2 \\ 2 & 0 & -2 \end{bmatrix}, \quad M_{kof}^T = \begin{bmatrix} 10 & -6 & 2 \\ -4 & 0 & 0 \\ 6 & -2 & -2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{\det M} \cdot M_{kof}^T = \frac{-1}{8} \begin{bmatrix} 10 & -6 & 2 \\ -4 & 0 & 0 \\ 6 & -2 & -2 \end{bmatrix} = \begin{bmatrix} -5/4 & 3/4 & -1/4 \\ 1/2 & 0 & 0 \\ -3/4 & 1/4 & 1/4 \end{bmatrix}$$

$$X = C \cdot (AB)^{-1} = [0 \ 4 \ 4] \cdot \left(-\frac{1}{8}\right) \begin{bmatrix} 10 & -6 & 2 \\ -4 & 0 & 0 \\ 6 & -2 & -2 \end{bmatrix} = \left(-\frac{1}{8}\right) [8 \ -8 \ -8]$$

$$X = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \text{ rješenje matricne jednačine}$$